Prediction of Bubble Terminal Velocities in Contaminated Water

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The rise of bubbles in liquids has been the focus of both academic and practical interests. It is determined, in addition to hydrodynamics, by interfacial physics and chemistry, and has attracted the attention of scientists for decades (Hadamard, 1911; Rybczynski, 1911; Boussineq, 1913; Levich, 1962). In engineering practice, bubbles are significant to bubble columns and inverse fluidized beds, including those employed in the flotation process of minerals and coal. Knowledge of the fundamentals of the bubble motion behavior, particularly the bubble rising velocity, is of great importance to understanding the characteristics of such processes.

It is well established that the presence of surface-active agents (surfactants) has profound effects on bubble motion and bubble terminal velocity through water (Gorodetskaya, 1949; Levich, 1962; Sam et al., 1996). Air bubbles behave as solid spheres until the bubble Reynolds number exceeds a critical value although bubbles remain spherical up to a much bigger Reynolds number. For bubbles in distilled water and in surfactant solutions, the critical value is about 40 and 130, respectively (Rosenberg, 1950; Fuerstenau and Wayman, 1958). Upon further increase in the Reynolds number, the drag coefficient of bubbles deviates from the standard drag curve of solid particles variously. For example, the recalculation based on the available experimental data in the literature by Karamanev (1994) shows that bubble drag coefficient is almost constant (0.95) when the bubble Reynolds number exceeds 130. It should be noted that the data used in the recalculation refers to the "contaminated" water. This situation commonly occurs in engineering processes, and Karamanev's findings can substantially simplify the prediction of bubble rise velocity for engineering purposes. A semi-analytical prediction of bubble terminal velocity was proposed by Karamanev (1994):

$$U = \sqrt{\frac{8g}{6^{2/3}\pi^{1/3}C}}\, V^{1/6} a \cdot Ta^{b}, \tag{1}$$

where g is the acceleration of gravity, V is the bubble volume, C is the drag coefficient. Ta is the Tadaki number and is related to the bubble Reynolds number Re and the Morton number M by

$$Ta = ReM^{0.23}. \tag{2}$$

The bubble Reynolds number is based on the diameter of volume-equivalent sphere D, which is independent of the bubble shape

$$Re = \frac{UD\delta}{\mu}, \tag{3}$$

where δ and μ are the liquid density and viscosity, respectively. The Morton number is defined by

$$M = \frac{g\mu^4}{\delta\sigma^3}, \tag{4}$$

where σ is the liquid surface tension. The numerical constants in Eq. 1 depend on the Tadaki number: $a = 1$, $b = 0$ when $Ta \le 2.11$ (spherical bubble); $a = 1.14$, $b = -0.176$ when $2.11 \le Ta \le 5.46$ (ellipsoidal bubble); $a = 1.36$, $b = -0.28$ when $5.46 \le Ta \le 16.53$ (ellipsoidal bubble); $a = 0.62$, $b = 0$ when $16.53 \le Ta$ (spherical cap bubble). Here, the constraints for Ta are resolved, allowing the continuity of the "aspect" ratio D/L between the subranges. They are therefore slightly different from that originally given by Tadaki and Maeda (1961).

A critical point of the prediction based on Eq. 1 is that both the drag coefficient and the Tadaki number depend on the bubble velocity. For the drag coefficient in Eq. 1 in the region $Re < 130$ Karamanev (1994) used the correlation of Turton and Levenspiel (1986): $C = 24(1 + 0.173Re^{0.657})\,Re + 0.413/(1 + 16{,}300Re^{-1.09})$. Alternatively, the author suggested that any correlation of the standard drag curve could be used instead of that of Turton and Levenspiel. Unfortunately, the correlation depends nonlinearly on the Reynolds number by rather complicated functions. Inserting the drag coefficient and the Tadaki number into Eq. 1 yields a highly nonlinear equation for the bubble velocity. An iterative procedure or a

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numerical computational technique is therefore required to determine bubble terminal velocity. Such a procedure is not convenient, especially when the bubble rise velocity is not the ultimate aim of engineering calculation; instead, an analytical prediction for the bubble rise velocity is often required. For example, the dependence of the bubble rise velocity on the bubble diameter is needed in the analysis of the predictions for the encounter probability between a solid particle and a bubble in flotation theory (Nguyen, 1994).

This article aims at a more straightforward and simple prediction for bubble terminal velocity, based on the experimental correlation between the drag coefficient and the Reynolds number which was presented by Karamanev (1994). The Karamanev analysis is strictly valid for bubbles in contaminated liquids, the "solid" bubbles. We are concerned with bubble rise in water. Therefore, the restrictions, namely, $790 < \delta < 1{,}350$ kg/m^3, $8\times10^{-4} < \mu < 0.12$ N/s/m^2, $0.02 < \sigma < 0.487$ N/m and $3.6\times10^{-14} < M < 3\times10^{-2}$ in the Karamanev analysis for drag coefficient are easy to meet. The presence of heteropolar reagents often has a strong effect on bubble rise velocity. Of these heteropolar reagents, surfactants such as frothing agents, which are strongly adsorbed on an air-water interface in an insoluble adsorption layer with high surface pressure, have the greatest influence on bubble terminal velocity (Harper, 1972). The contamination of water concerned in this article is due to frothers. The "solid" bubble requirement can be therefore easily satisfied. Many frothers typically used in mineral flotation, added to water by a very small concentration (few ppm), sufficiently render constant bubble terminal velocity in water (Sam et al., 1996). The authors also show that bubble rising velocity passes many stages and terminal velocity can be reached by a sufficiently tall water column. Under this circumstance, we recall the vast experimental data for "terminal" velocity reported in the literature. There is another circumstance, which could affect the experimental data for the terminal velocity, for example, the wall effect. In this context, the results reported in this article are valid for single bubbles rising through extended liquids.

Terminal Velocity vs. Archimedes Number

Many explicit equations have been reported in the literature to predict particle velocity, based on the Archimedes number $Ar = D^3\delta^2 g/\mu^2$ (also known, in slightly different expression, as the Best number or the Galileo number or the Castleman function). The first group of these equations is derived from the correlation of the particle Reynolds number with the Archimedes number. In the second group, the terminal velocity is directly derived from the correlation between the Lyashchenko number $Ly = U^3\delta/g/\mu$ (also known as the dimensionless terminal velocity) and the Archimedes number. Using the second method, a simple and unified formula for terminal velocities of solid spheres was predicted (Nguyen et al., 1997). This prediction covers a broad range of data from zero to 1,000 for the Reynolds number. In this article, we extend this approach to the problem of terminal velocity of "solid" bubbles.

The shape factor has not been considered in calculation of drag coefficient of gas bubbles by most authors. Since gas bubbles are deformable, it is essential to involve the shape influence in predicting bubble terminal velocities. Following the generic definition of drag force F, we have

$$C = \frac{2F}{A\delta U^2}, \tag{5}$$

where A is the bubble cross section area projected perpendicularly to its path. Suppose that L is the diameter of the "projected area." The force balance at the bubble terminal state yields

$$C = \frac{4Dg}{3U^2}\left(\frac{D}{L}\right)^2. \tag{6}$$

Inserting the Tadaki prediction for D/L (Tadaki and Maeda, 1961) into Eq. 6, we obtain

$$C = \frac{8a^2Dg}{3U^2}M^{0.46b}Re^{2b}. \tag{7}$$

We define the following modified Archimedes number and modified Lyashchenko number

$$Ar^* = \frac{3\,Re^{2-2b}C}{4} \tag{8}$$

$$Ly^* = \frac{4\,Re^{2b+1}}{3C} \tag{9}$$

It follows that the modified Archimedes number is dependent on the bubble equivalent diameter and independent of the bubble terminal velocity

$$Ar^* = D^3\frac{\delta^2 g}{\mu^2}(a^2M^{0.46b}) \tag{10}$$

The modified Lyashchenko number is, on the other hand, dependent solely on the bubble terminal velocity

$$Ly^* = U^3\frac{\delta}{g\mu}\frac{1}{a^2M^{0.46b}}. \tag{11}$$

Equations 8 and 9 enable us, on the basis of the available dependence of the drag coefficient on the Reynolds number, to calculate the modified Archimedes number and Lyashchenko number and to find out the correlation between them. Equations 10 and 11 can be used to predict the terminal velocity, based on the Ly^*-Ar^* correlation or vice versa.

Prediction of Bubble Terminal Velocities When $Re < 130$

When $Re < 130$, the Tadaki number of bubble rise in contaminated water may be smaller than 2. The shape of bubbles is spherical. In this regime, bubble drag coefficient is equal to that of solid particles. The following correlation between the bubble Lyashchenko number Ly and the bubble Archimedes number Ar can be found (Nguyen et al., 1997)

$$(Ly)^{1/3} = \frac{(Ar)^{2/3}}{18^3} \frac{1}{1 + \dfrac{Ar/96}{(1+0.079\,Ar^{0.749})^{0.755}}}. \quad (12)$$

We immediately obtain the following prediction for bubble terminal velocity in the regime $Re < 130$

$$U = U_s \left\{ 1 + \frac{Ar/96}{(1+0.079\,Ar^{0.749})^{0.755}} \right\}^{-1}, \quad (13)$$

where $U_s = D^2 g\delta/18/\mu$, and is the bubble terminal velocity predicted by Stokes' law. The factor on the righthand side in Eq. 13 accounts for the deviation of terminal velocity of rigid bubbles from the Stokes equation. When Ar is small, the factor rapidly converges to unity and Eq. 13 transforms to the Stokes equation for small bubbles.

Bubble Terminal Velocities at Reynolds Numbers Higher than 130

In this regime, drag coefficient of "solid" bubbles based on the projected area can be considered as constant $C = k$, as shown by Karamanev and Nikolov (1992) and Karamanev (1994). Following these authors, we take the value $k = 0.95$. Elimination of bubble Reynolds numbers from Eqs. 8 and 9 yields

$$Ly^* = \frac{4}{3k} \left\{ \frac{4\,Ar^*}{3k} \right\}^{(2b+1)/(2-2b)} \quad (14)$$

Substituting Eqs. 10 and 11 into Eq. 14, we obtain the following formula for bubble terminal velocity based on the bubble Archimedes number and the Morton number

$$U = \sqrt[3]{\frac{g\mu}{\delta}} \left\{ \frac{4a^2 M^{0.46b}}{3k} \right\}^{1/(2-2b)} (Ar)^{(2b+1)/(6-6b)}. \quad (15)$$

In the case of spherical cap bubbles, $b = 0$ and $a = 0.62$ or $a\sqrt{3} \cong 1$ (Tadaki and Maeda, 1961). Since $\sqrt{k} \cong 1$, Eq. 15 transforms to, under this circumstance

$$U = \frac{2}{3}\sqrt{Dg}. \quad (16)$$

This is exactly the celebrated Davies and Taylor (1950) equation.

Constraints for the Terminal Velocity Predictions, Based on the Bubble Archimedes Number and the Morton Number

The bubble rise velocities given by Eqs. 13 and 15 are developed by using the bubble Reynolds number and the Tadaki number conditions. We need to convert them to the conditions based on the bubble Archimedes number and the Morton number since the bubble Reynolds numbers are not known in advance. Inserting the terminal velocities given by Eqs. 13 and 15 into Eq. 3 we obtain the following expressions for the bubble Reynolds number, respectively

$$Re = \frac{Ar}{18} \left\{ 1 + \frac{Ar/96}{(1+0.079\,Ar^{0.749})^{0.755}} \right\}^{-1} \quad (17)$$

$$Re = \left\{ \frac{4a^2 M^{0.46b} Ar}{3k} \right\}^{1/(2-2b)}. \quad (18)$$

Inserting Eq. 18 into Eq. 2 yields

$$Ta = \left\{ \frac{4a^2 M^{0.46} Ar}{3k} \right\}^{1/(2-2b)}. \quad (19)$$

The prediction given by Eq. 13 is developed for air bubbles whose drag coefficient is expected to be equal to the drag coefficient of solid spheres. As previously mentioned, the drag of bubble rise in contaminated water follows that of solid spheres until the bubble Reynolds number exceeds 130. To date, this critical value has been given approximately. However, here, we can predict this "transition" value precisely. The predictions given by Eqs. 17 and 18 must give the same value for the bubble Reynolds number at the point of the deviation of the bubble drag coefficient from that of solid spheres. This can ocur in the region of spherical bubbles, that is, $a = 1$, $b = 0$ and $Ta \leq 2.11$. Applying the first conditions to Eqs. 17 and 18 yields $Ar = 12{,}332$. Inserting this value for Ar into Eqs. 17 and 19 we obtain, respectively, $Re = 132$ and $Ta = 131.56\ M^{0.23}$. The obtained bubble Reynolds number is close to the approximately given value 130 and acceptable. The condition $Ta \leq 2.11$ yields $M \leq 1.57 \times 10^{-8}$, which can be satisfied by most contaminated water systems.

The Ar constraints for a and b can be found by employing Eq. 19

$$Ar = M^{-0.46} \frac{3kTa^{2-2b}}{4a^2}. \quad (20)$$

When the boundaries of Ta between the subranges, rounded off to four significant figures, are used, we obtain the constraints for a and b as given in Table 1.

Comparison with the Experimental Data and Discussion

The experimental data for bubble terminal velocity in water is available in numerous reports. The prediction for the bubble velocity in this article is developed using the bubble

Table 1. Numerical Parameters a and b (Eq. 15) in Dependence on Bubble Archimedes Number and Morton Number

$12{,}332 \leq Ar \leq 3.158\ M^{-0.46}$	$a = 1$, $b = 0$
$3.158\ M^{-0.46} \leq Ar \leq 29.654\ M^{-0.46}$	$a = 1.14$, $b = -0.176$
$29.654\ M^{-0.46} \leq Ar \leq 506.719\ M^{-0.46}$	$a = 1.36$, $b = -0.28$
$506.719\ M^{-0.46} \leq Ar$	$a = 0.62$, $b = 0$

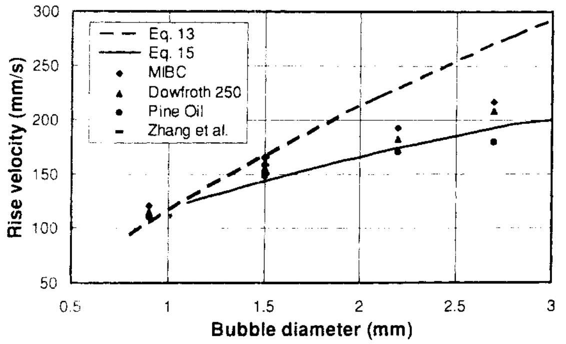

Figure 1. Predictions by Eqs. 13 and 15 (curves) vs. available experimental data for bubble rise velocity in frother-contaminated water (points).

The diamonds, triangles and filled circles describe the experimental data reported by Sam et al. (1996). The experimental data reported by Zhang et al. (1996) are for bubbles of 1.5 mm diameter. MIBC is methyl isobutyl carbinol. Dowfroth is a commercial frother used in mineral flotation, based on methoxy polypropylene glycols.

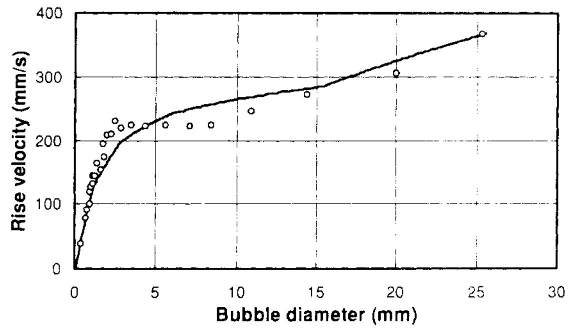

Figure 2. Predictions by Eqs. 13 and 15 (curves) vs. Haberman and Morton's (1953) data for bubble rise velocity in tap water (points).

drag coefficient recalculated by Karamanev (1994). In Karamanev's analysis, the available experimental data with liquids considered as contaminated is employed. To justify whether the liquids are contaminated, the author used the simple criterion proposed by Clift et al. (1978): in contaminated water, the velocity-bubble volume curve should not pass through a maximum peak. Another circumstance we focus on is whether the terminal (constant) bubble rise velocity is reported in the available literature. We found that the data recently reported by Sam et al. (1996) and Zhang et al. (1996) is satisfactory. The comparison between the predictions in this article and the experimental data is illustrated in Figure 1. Although a good agreement can be observed, the measured data is a bit higher than the predicted velocities.

It is noteworthy that for frother-contaminated water, the Morton number is often within the range of 10^{-11} to 10^{-9}. Since the diameter of bubbles typically used in flotation is often smaller than 2.5 mm, the condition $Ar \leq 3.158\ M^{-0.46}$ is satisfied. In this case, Eq. 15 reduces to

$$U = \sqrt{\frac{4Dg}{3}}, \qquad (21)$$

which is $\sqrt{3}$ times higher than the velocity predicted by the famous Davies and Taylor (1950) equation. Equations 15 and 21 are expected to be useful for the prediction of bubble rise velocity in the mineral flotation process.

Pure systems and contaminated systems

There is evidence in the vast literature on the experimental data of bubble rise velocity that drag and rise velocity of bubbles in water are rather sensitive to water purity. Even though the amount of impurity may be so small that there is no measurable change in the bulk water properties, a contaminant can resist circulation inside an air bubble, thereby significantly increasing the drag and dramatically reducing bubble rise velocity. Contaminated water is so ordinary and its purifying is so stringent that the study of bubble rise in pure water systems is often a problem (even distilled water tends to contain sufficient impurity to prevent circulation in small bubbles). At relatively low contaminant concentrations, as low as few ppm in the case of frothers, the bubble rise velocity becomes nearly independent of surfactant concentration (Sam et al., 1996). The term "contaminated water" in this article refers to these cases and is applicable to most systems of engineering importance. Practically, it can be satisfied by the simple criterion of Clift et al. (1978) mentioned in the first paragraph of this section. The contaminant effect is the most profound for air bubbles since an air-water interface exhibits relatively high interfacial tension and an extremely low ratio of viscosity of the bulk phases. Contaminated water is not a serious limitation in engineering practice. Figure 1 exemplifies our prediction for bubble rise in frother-contaminated water. Here, we compare the prediction with Haberman and Morton's (1953) data on tap water. The result is given in Figure 2. The agreement is similar to the agreement found for frothing agents. This might support our model in the broad range of "contaminated water." In systems of intermediate purity, we cannot negate the role of interface physics and chemistry dealing with surfactant adsorption, transport, and desorption. However, the model equations are so complicated and nonlinear that the science needs a numerical computation for predicting bubble rise in water (McLaughlin, 1996). It is expected that the simple prediction given by Eqs. 13 and 15 for bubble rise velocity based on experimental data is practically useful, in particular, for anyone who needs the prediction to explore the underlying process principles in engineering.

Dimensionless-parameter group considerations

Both the density and viscosity of air are so small compared with those of water that they have been found to be unimportant in determining the bubble rise velocity. Therefore, it is often sufficient to consider only two dimensionless groups, the Eötvos number Eo ($= gD^2\delta/\sigma$ and sometimes called the Bond number), and the Morton number in a correlation for bubble rise velocity. The bubble rise velocity prediction in this article is expressed by the Archimedes number and the Morton number. Since $(Ar)^2 = (Eo)^3/M$, the bubble rise velocity in this article can be easily converted to the correlation by the Eötvos number and the Morton number. The key of the present correlation is the bubble aspect ratio in Eq. 6. The amount of bubble deformation is usually presented by the Weber number $We = U^2D\delta/\sigma$, which occurs in the stress balance on the bubble surface in the normal direction. Only a few models for bubble deformation have been theoretically derived. For low Re and small deformations, Taylor and Acrivos' (1964) theory yields $L/D = 1 + (5We/96)$. For high Re and small deformations, Moore's (1959) boundary theory predicts that $E = 1 + (9We/64) + O(We^2)$ where E is the eccentricity defined as the ratio of the cross-stream axis to the parallel axis. When the bubble is an oblate spheroid $L/D = E^{1/3}$. A high-order correlation for E was also given by Moore (1965). The use of these theoretical predictions for the bubble aspect ratio in determining bubble rise velocity has a limitation since the Weber number depends on the rise velocity. One may use the empirical correlation for eccentricity $E = 1 + 0.129Eo$ (Wellek et al., 1966). This correlation was found for $Eo < 40$ and $M < 10^{-6}$. The use of Wellek et al.'s correlation for determination of bubble rise velocity is possible only when the bubble deformation is spheroidal. If the bubble deformation is strong, the spheroidal shape is lost. The eccentricity E is difficult to be converted to the bubble aspect ratio L/D. It can be seen that the empirical correlation for the bubble aspect ratio given by Tadaki and Maeda (1961) is really a useful tool for determining the bubble rise velocity. It covers all regimes of bubble deformation, from spheroidal bubbles to bubble caps. Experimental data show that deformation of bubbles and drops depends on Eo and M (Wellek et al., 1966; Tsuge and Hibino, 1977). However, for contaminated systems of low M ($M \leq 10^{-6}$) such as air bubbles in water, the eccentricity E was found to be independent of M (Wellek et al., 1966). The experimental systems in Tadaki and Maeda's (1961) measurements are contaminated water of low $M \leq 10^{-6}$, except one system consisting of 75% glycerin and water. We conclude at this point that the range of applicability of the expressions for bubble aspect ratio used in this article is $M \leq 10^{-6}$.

Conclusions

Terminal bubble velocity in contaminated water is described by Eqs. 13 and 15. The first equation can be used with the bubbles with the Archimedes number not greater than 12332. Over this range, the second equation is valid with the specific conditions for the numerical constants given in Table 1. The range of applicability of the bubble rise velocity predictions and the expressions in this article is $M \leq 10^{-6}$. Contaminated water considered in this article is justified by the simple criterion: the velocity-bubble volume curve should not pass through a maximum peak. The predictions given in this article are more straightforward than those given by Karamanev (1994) and do not require numerical computation techniques to determine bubble terminal velocity. Thus, the predictions are useful for anyone needing explicit formulas for bubble rise velocity to investigate the underlying phenomena of engineering processes. For bubbles typically used in flotation, Eq. 15 is simplified to Eq. 21, which predicts bubble rising terminal velocity $\sqrt{3}$ times higher than the velocity predicted by the famous Davies and Taylor equation for spherical bubble caps.

Acknowledgments

The author would like to express his sincere thanks to Professor J. A. Finch, McGill University, who made the experimental results of his group available. Professor John Ralston, University of South Australia, is gratefully acknowledged for support provided to the author during his stay in Adelaide where this work was performed. The author is grateful to an anonymous reviewer for the digitized Haberman and Morton's data and the copy of the report.

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Manuscript received Apr. 21, 1997, and revision received Sept. 2, 1997.